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# Thermal entanglement in the anisotropic Heisenberg XXZ model with the Dzyaloshinskii–Moriya interaction

**Da-Chuang Li[1], Xian-Ping Wang[1] and Zhuo-Liang Cao[1,2,3]**

[1] Key Laboratory of Opto-electronic Information Acquisition and Manipulation
(Ministry of Education), School of Physics and Material Science, Anhui University,
Hefei 230039, People's Republic of China
[2] Department of Physics, Hefei Teachers College, Hefei 230061, People's Republic of China

E-mail: dachuang@ahu.edu.cn and zhuoliangcao@gmail.com

## Abstract
The thermal entanglement is investigated in a two-qubit Heisenberg XXZ system with
Dzyaloshinskii–Moriya (DM) interaction. It is shown that the entanglement can be efficiently
controlled by the DM interaction parameter and coupling coefficient $J_z$. $D_x$ (the $x$-component
parameter of the DM interaction) has a more remarkable influence on the entanglement and the
critical temperature than $D_z$ (the $z$-component parameter of the DM interaction). Thus, by
changing the DM interaction direction, we can get a more efficient control parameter to increase
the entanglement and the critical temperature.

## 1. Introduction

Entanglement has been extensively studied in recent years
because it has the fascinating nonclassical nature of quantum
mechanics, and it plays a key role in quantum information
processing [1, 2]. The quantum entanglement in solid
state systems such as spin chains is an important emerging
field [3–8]: spin chains are natural candidates for the
realization of entanglement and spin has been researched in
many other systems, such as superconductors [9, 10], quantum
dots [11–13] and trapped ions [14, 15].

In order to characterize qualitatively and quantitatively
the entanglement properties of condensed matter systems and
apply them in quantum information, the thermal entanglement
qualities in the Heisenberg model have been extensively
studied [7, 8, 16, 17] and many schemes of teleportation
via thermal entangled states have been reported [18–21]. In
condensed matter systems, the Heisenberg chains have also
been used to construct a quantum computer [22], perform
quantum computation [23–25], etc.

In those studies the spin–spin interaction was considered,
but the spin–orbit coupling was rarely considered. In
particular, the influences of the $x$-component DM interaction

parameter on the entanglement and the critical temperature
have never been reported. In this paper we investigate the
influence of the Dzyaloshinskii–Moriya interaction parameter
(arising from the spin–orbit coupling) and coupling coefficient
$J_z$ on the entanglement of a two-qubit anisotropic Heisenberg
XXZ spin chain. We show that the DM interaction
parameter and the coupling coefficient $J_z$ are both efficient
control parameters of entanglement. Increasing them can
enhance the entanglement or slow down the decrease of
the entanglement. In addition, by analyzing how we know
that different component parameters of the DM interaction
have different influences on the entanglement and the critical
temperature $T_c$, the parameter $D_x$ ($x$-component parameter
of the DM interaction) has a more remarkable influence than
the parameter $D_z$ (the $z$-component parameter of the DM
interaction). So a more efficient control parameter can be
obtained by changing the DM interaction direction.

Our paper is organized as follows. In section 2,
we introduce the Hamiltonian of the two-qubit anisotropic
Heisenberg XXZ chain with the $z$-component parameter of the
DM interaction, calculate the concurrence of this system and
analyze the influence of parameters on the entanglement in
the ground state and thermal state. In section 3, we similarly
analyze the model of the two-qubit Heisenberg XXZ chain with
the $x$-component parameter of the DM interaction. Then we

[3] Author to whom any correspondence should be addressed.

compare the influences of the two component parameters of the DM interaction on the entanglement in section 4. Finally, in section 5 a discussion concludes the paper.

## 2. XXZ Heisenberg model with DM interaction parameter $D_z$

The Hamiltonian $H$ for a two-qubit anisotropic Heisenberg XXZ chain with DM interaction parameter $D_z$ is

$$H = J\sigma_1^x\sigma_2^x + J\sigma_1^y\sigma_2^y + J_z\sigma_1^z\sigma_2^z + D_z(\sigma_1^x\sigma_2^y - \sigma_1^y\sigma_2^x), \quad (1)$$

where $J$ and $J_z$ are the real coupling coefficients, $D_z$ is the $z$-component parameter of the DM interaction and $\sigma^i$ ($i = x, y, z$) are Pauli matrices. The coupling constants $J > 0$ and $J_z > 0$ correspond to the antiferromagnetic case, while $J < 0$ and $J_z < 0$ correspond to the ferromagnetic case. This model is reduced to the isotropic XX model when $J_z = 0$ and to the isotropic XXX model when $J_z = J$. Parameters $J$, $J_z$ and $D_z$ are dimensionless.

In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the Hamiltonian (1) can be expressed as

$$H = \begin{pmatrix} J_z & 0 & 0 & 0 \\ 0 & -J_z & 2J + 2iD_z & 0 \\ 0 & 2J - 2iD_z & -J_z & 0 \\ 0 & 0 & 0 & J_z \end{pmatrix}. \quad (2)$$

By a straightforward calculation we can obtain $H$ eigenstates:

$$|\Phi_1\rangle = |00\rangle, \quad (3a)$$

$$|\Phi_2\rangle = |11\rangle, \quad (3b)$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{2}}(e^{i\theta}|01\rangle + |10\rangle), \quad (3c)$$

$$|\Phi_4\rangle = \frac{1}{\sqrt{2}}(e^{i\theta}|01\rangle - |10\rangle), \quad (3d)$$

with corresponding eigenvalues:

$$E_1 = J_z, \quad (4a)$$

$$E_2 = J_z, \quad (4b)$$

$$E_3 = -J_z + 2w, \quad (4c)$$

$$E_4 = -J_z - 2w, \quad (4d)$$

where $w = \sqrt{J^2 + D_z^2}$ and $\theta = \arctan(\frac{D_z}{J})$.

The state of a spin chain system at thermal equilibrium is $\rho(T) = \frac{\exp(-\beta H)}{Z}$, where $Z = \text{tr}[\exp(-\beta H)]$ is the partition function of the system, $H$ is the system Hamiltonian and $\beta = \frac{1}{K_B T}$, with $T$ temperature and $K_B$ the Boltzmann constant which we take equal to 1 for simplicity. Here $\rho(T)$ represents a thermal state, so the entanglement in the thermal state is called thermal entanglement [26]. In the above standard basis, the state of this system at thermal equilibrium can be expressed as

$$\rho(T) = \frac{1}{Z}\begin{pmatrix} e^{-\beta J_z} & 0 & 0 & 0 \\ 0 & u & ve^{i\theta} & 0 \\ 0 & ve^{-i\theta} & u & 0 \\ 0 & 0 & 0 & e^{-\beta J_z} \end{pmatrix}, \quad (5)$$

where $u = \frac{1}{2}(1 + e^{4\beta w})e^{\beta(J_z - 2w)}$, $v = \frac{1}{2}(1 - e^{4\beta w})e^{\beta(J_z - 2w)}$ and $Z = 2e^{-\beta J_z}[1 + e^{2\beta J_z}\cosh(2\beta w)]$.

In what follows, we consider the concurrence to quantify the amount of entanglement of the above two-qubit system state $\rho(T)$. The concurrence [27, 28] is defined as $C(\rho(T)) = \max[2\max(\lambda_i) - \Sigma_i\lambda_i, 0]$, where $\lambda_i$ ($i = 1, 2, 3, 4$) are the square roots of the eigenvalues of the matrix $R = \rho S\rho^* S$, in which $S = \sigma_1^y \bigotimes \sigma_2^y$, $\rho$ is the density matrix of equation (5) and the asterisk denotes the complex conjugate. After some straightforward calculation, we get

$$\lambda_1 = \frac{1}{Z}e^{-\beta J_z}, \quad (6a)$$

$$\lambda_2 = \frac{1}{Z}e^{-\beta J_z}, \quad (6b)$$

$$\lambda_3 = \frac{1}{Z}e^{\beta(J_z - 2w)}, \quad (6c)$$

$$\lambda_4 = \frac{1}{Z}e^{\beta(J_z + 2w)}, \quad (6d)$$

thus the corresponding concurrence can be expressed as

$$C(\rho(T)) = \max\left\{\frac{e^{\beta J_z}}{Z}\big(|e^{2\beta w} - e^{-2\beta J_z}| \right.$$
$$\left. - e^{-2\beta w} - e^{-2\beta J_z}\big), 0\right\}. \quad (7)$$

The concurrence is invariant under the substitutions $J \to -J$ and $D_z \to -D_z$, so we can restrict $J > 0$ and $D_z > 0$ without loss of generality. The concurrence ranges from 0 to 1, $C(\rho(T)) = 0$ and $C(\rho(T)) = 1$ indicate the vanishing entanglement and the maximal entanglement, respectively. We can see from equation (7) that the entanglement $C(\rho(T)) = \frac{e^{\beta J_z}}{Z}(e^{2\beta w} - e^{-2\beta J_z} - e^{-2\beta w} - e^{-2\beta J_z})$ if $J_z > -w$, and $C(\rho(T)) = 0$ if $J_z < -w$. Here we analyze the $J_z > -w$ case.

When $T = 0$, the system is in its ground state. It is easy to find that the ground-state energy is equal to

$$E_4 = -J_z - 2w, \qquad \text{if } J_z > -w, \quad (8a)$$

$$E_1 = E_2 = J_z, \qquad \text{if } J_z < -w. \quad (8b)$$

Thus, the ground state is the disentangled state $|\Phi_1\rangle$ or $|\Phi_2\rangle$ when $J_z < -w$, and the ground state is the entangled state $|\Phi_4\rangle$ when $J_z > -w$. The entanglement of the ground state $|\Phi_4\rangle$ is the maximal entanglement with $C(|\Phi_4\rangle) = 1$.

As the temperature increases the thermal fluctuation will be introduced into the system. Thus the entanglement will be changed due to the mix of the ground states and the excited states. When the temperature is higher than a critical temperature the entanglement is zero. Quantum phase transition happens at the critical temperature $T_c$. From equation (7), we obtain the following critical temperature equation:

$$e^{\frac{2J_z}{T_c}}\sinh\left(\frac{2w}{T_c}\right) = 1. \quad (9)$$

To see the change of the entanglement in detail, we analyze the concurrence of equation (7). By fixing some
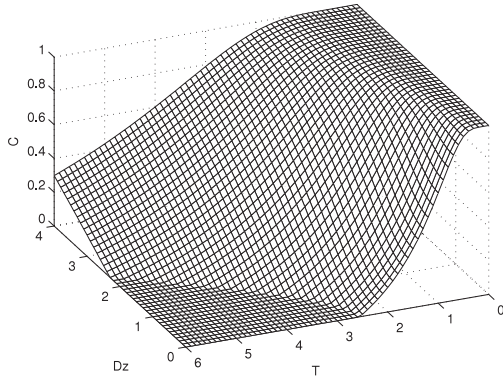
**Figure 1.** The concurrence is plotted versus $T$ and $D_z$ where the coupling constants $J = 1$ and $J_z = 0.2$.
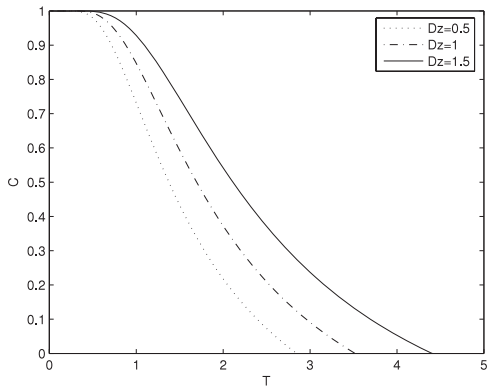


**Figure 2.** The concurrence is plotted as a function of the temperature $T$ for different $D_z$. Here $J = 1$ and $J_z = 0.2$.

parameters we can know the roles of the other parameters and the variation of the entanglement. In figure 1, the thermal entanglement is plotted versus $T$ and $D_z$ where the coupling constants $J = 1$ and $J_z = 0.2$. From the figure, it is obvious that the increased temperature $T$ can decrease the entanglement. The reason is the mixing of the maximally entangled state with other states. In addition, it is easy to see that the entanglement will increase as the DM interaction parameter $D_z$ increases. Figure 2 demonstrates the concurrence versus temperature for different DM coupling parameters $D_z$ when $J = 1$ and $J_z = 0.2$. It shows that the concurrence will decrease with increasing temperature $T$ and increase with increasing $D_z$ for a certain temperature. The critical temperature $T_c$ determined by equation (9) is dependent on $D_z$. Increasing $D_z$ can increase the critical temperature above which the entanglement vanishes. Similarly, figure 3 shows the concurrence versus temperature for different anisotropic coupling parameters $J_z$ when $J = 1$ and $D_z = 1$. We can see that the entanglement decreases with the increase of temperature, and by increasing $J_z$, the critical temperature is increased and the entanglement is enhanced for a certain temperature.

So the DM interaction parameter $D_z$ and anisotropic coupling coefficient $J_z$ are both efficient control parameters of entanglement. By increasing them, we can enhance the
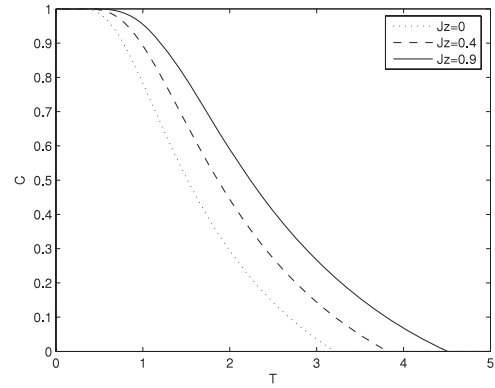


**Figure 3.** The concurrence versus $T$ for different $J_z$ in the system with $z$-component parameter $D_z$. Here $J = 1$ and $D_z = 1$.

entanglement or increase the critical temperature to slow down the decrease of the entanglement.

## 3. XXZ Heisenberg model with DM interaction parameter $D_x$

Here we consider the case of the two-qubit anisotropic Heisenberg XXZ chain with DM interaction parameter $D_x$. The Hamiltonian is

$$H' = J\sigma_1^x\sigma_2^x + J\sigma_1^y\sigma_2^y + J_z\sigma_1^z\sigma_2^z + D_x(\sigma_1^y\sigma_2^z - \sigma_1^z\sigma_2^y), \quad (10)$$

where $D_x$ is the $x$-component parameter of the DM interaction, and $J$, $J_z$ and $\sigma^i$ ($i = x, y, z$) are the same as in section 2. Parameters $D_x$, $J$ and $J_z$ are dimensionless.

In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the Hamiltonian (10) can be rewritten as

$$H' = \begin{pmatrix} J_z & iD_x & -iD_x & 0 \\ -iD_x & -J_z & 2J & iD_x \\ iD_x & 2J & -J_z & -iD_x \\ 0 & -iD_x & iD_x & J_z \end{pmatrix}. \quad (11)$$

After a straightforward calculation we obtain $H'$ eigenstates:

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad (12a)$$

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad (12b)$$

$$|\Psi_3\rangle = \frac{1}{\sqrt{2}}(-i\sin\phi|00\rangle + \cos\phi|01\rangle - \cos\phi|10\rangle + i\sin\phi|11\rangle), \quad (12c)$$

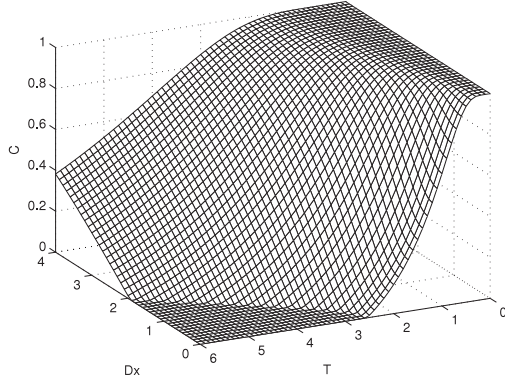$$|\Psi_4\rangle = \frac{1}{\sqrt{2}}(-i\sin\varphi|00\rangle + \cos\varphi|01\rangle - \cos\varphi|10\rangle + i\sin\varphi|11\rangle), \quad (12d)$$

with corresponding eigenvalues

$$E_1' = J_z, \quad (13a)$$

$$E_2' = 2J - J_z, \quad (13b)$$

$$E_3' = -J + w', \quad (13c)$$

**Figure 4.** The concurrence is plotted versus $T$ and $D_x$ where the coupling constants $J = 1$ and $J_z = 0.2$.



**Figure 5.** The concurrence versus $T$ for different $J_z$ in the system with $x$-component parameter $D_x$. Here $J = 1$ and $D_x = 1$.

$$E'_4 = -J - w', \qquad (13d)$$

where $\phi = \arctan(\frac{2D_x}{J+J_z-w'})$, $\varphi = \arctan(\frac{2D_x}{J+J_z+w'})$ and $w' = \sqrt{(J + J_z)^2 + 4D_x^2}$.

For convenience of analysis, we assume $J_z \leqslant J$ in this section (for the case $J_z > J$ we can get some similar results). Here, it is easy to see that the system's ground-state energy is $E'_4 = -J - w'$. Thus the corresponding ground state $|\Psi_4\rangle$ is the maximally entangled state with $C(|\Psi_4\rangle) = 1$. In fact the four eigenstates in equation (12) are all maximally entangled states. This phenomenon indicates that the ground states have more entanglement in this system than the system of equation (1).

At thermal equilibrium the density matrix of this two-qubit spin chain system has the following form:

$$\rho'(T) = \frac{\exp(-\beta H')}{Z'} = \frac{1}{2Z'} \begin{pmatrix} \mu_+ & -\xi & \xi & \mu_- \\ \xi & \nu_+ & \nu_- & -\xi \\ -\xi & \nu_- & \nu_+ & \xi \\ \mu_- & \xi & -\xi & \mu_+ \end{pmatrix}, \qquad (14)$$

where $Z' = 2e^{-\beta J} \cosh[\beta(J - J_z)] + 2e^{\beta J} \cosh(\beta w')$ is the partition function of the system, $H'$ is the system Hamiltonian and $\beta = \frac{1}{K_B T}$ with the Boltzmann constant $K_B \equiv 1$. $\mu_\pm = e^{-\beta J_z} \pm (e^{\beta(J-w')} \sin^2 \phi + e^{\beta(J+w')} \sin^2 \varphi)$, $\nu_\pm = e^{\beta(J_z-2J)} \pm (e^{\beta(J-w')} \cos^2 \phi + e^{\beta(J+w')} \cos^2 \varphi)$, and $\xi = ie^{\beta(J-w')} \sin \phi \cos \phi + ie^{\beta(J+w')} \sin \varphi \cos \varphi$.
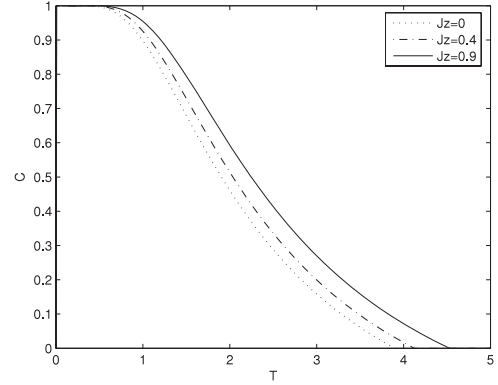
In what follows, we calculate the square roots of the eigenvalues of the matrix $R' = \rho' S \rho'^* S$, where $\rho'^*$ is the complex conjugate of $\rho'$ and $S = \sigma_1^y \bigotimes \sigma_2^y$. The square roots of the eigenvalues of the matrix $R'$ are

$$\lambda'_1 = \frac{1}{Z'} e^{\beta(J_z-2J)}, \qquad (15a)$$

$$\lambda'_2 = \frac{1}{Z'} e^{-\beta J_z}, \qquad (15b)$$

$$\lambda'_3 = \frac{1}{Z'} e^{\beta J} \left[ \cosh(\beta w') + \sqrt{\cosh^2(\beta w') - 1} \right], \qquad (15c)$$

$$\lambda'_4 = \frac{1}{Z'} e^{\beta J} \left[ \cosh(\beta w') - \sqrt{\cosh^2(\beta w') - 1} \right]. \qquad (15d)$$

According to the methods in [27, 28], when $J_z \leqslant J$ we obtain the corresponding concurrence:

$$
\begin{aligned}
&C(\rho'(T)) \\
&= \max \left\{ \frac{1}{Z'} \left[ \left| e^{\beta J} \left( \cosh(\beta w') + \sqrt{\cosh^2(\beta w') - 1} \right) \right. \right. \right. \\
&\quad \left. \left. - e^{-\beta J_z} \right| - e^{\beta J} \left( \cosh(\beta w') - \sqrt{\cosh^2(\beta w') - 1} \right) \\
&\quad \left. \left. - e^{\beta(J_z-2J)} \right], 0 \right\}.
\end{aligned}
\qquad (16)
$$

To analyze the role of a parameter and the variation of the entanglement, we restrict the parameters $D_x > 0$, $J_z > 0$ and $J > 0$. In figure 4, the concurrence is plotted versus $T$ and $D_x$ when the coupling constants $J = 1$ and $J_z = 0.2$. It is evident that increasing temperature will decrease the entanglement, and increasing $D_x$ will enhance the entanglement and increase the critical temperature $T'_c$, which is determined by equation (16). Figure 5 demonstrates the concurrence versus $T$ for different $J_z$ with $x$-component parameter $D_x = 1$ and $J = 1$. It is easy to find that increasing $J_z$ can increase the critical temperature and enhance the entanglement for a certain temperature. So $D_x$ and $J_z$ are both efficient control parameters of entanglement, too.

## 4. The comparison between the two DM interaction component parameters

From sections 2 and 3, we know that the $x$-component parameter $D_x$ and the $z$-component parameter $D_z$ of the DM interaction have similar qualities. They are both efficient control parameters of entanglement, so increasing them can enhance the entanglement or increase the critical temperature to slow down the decrease of entanglement.

In this section, we mainly analyze the differences between the $x$-component parameter and $z$-component parameter. We note that there is s smaller disentanglement region in figure 4 than in figure 1, where $J = 1$ and $J_z = 0.2$, and the temperature and the spin–orbit coupling parameter have the same range. We also see that increasing the $x$-component parameter $D_x$ can make the entanglement increase more rapidly. For example, when $T = 6$ the concurrence
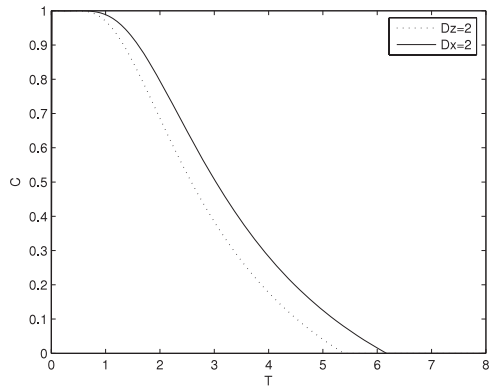
**Figure 6.** The concurrence is plotted as a function of the temperature $T$ for $D_z = 2$ and $D_x = 2$. Here $J = 1$ and $J_z = 0.2$.

increases more rapidly in figure 4 than in figure 1. These phenomena show that the $x$-component parameter $D_x$ has a more remarkable influence than the $z$-component parameter $D_z$. In figure 6, the concurrence is plotted as a function of the temperature $T$ for $D_z = 2$ and $D_x = 2$ with $J = 1$ and $J_z = 0.2$. It is easy to see that, for the same $D_x$ and $D_z$, the $x$-component parameter $D_x$ has a higher critical temperature and more entanglement for a certain temperature than the $z$-component parameter $D_z$. We show directly the differences between different component parameters of the DM interaction in figure 6. So for different directions of DM interaction, we can increase the entanglement and the critical temperature with different efficiencies.

## 5. Discussion

The thermal entanglement of a two-qubit Heisenberg XXZ system with DM interaction is investigated. The DM interaction parameter and coupling coefficient $J_z$ are efficient control parameters of the entanglement. By increasing the parameters, we can enhance the entanglement or increase the critical temperature to slow down the decrease of the entanglement. In addition, we have also investigated the differences between the $x$-component parameter $D_x$ and the $z$-component parameter $D_z$ of the DM interaction. Entanglement can be increased more rapidly by increasing $D_x$ more than $D_z$. When $D_x$ and $D_z$ have the same value, $D_x$ has a higher critical temperature $T_c$ than $D_z$. Thus, by changing the direction of the DM interaction, we can get a more efficient control parameter to increase the entanglement and the critical temperature.

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